A useful fourth moment matrix of a random vector

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A USEFUL FOURTH MOMENT MATRIX OF A RANDOM VECTOR

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Matrices

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A USEFUL FOURTH MOMENT MATRIX OF A RANDOM VECTOR

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Abstract

Let x be a real random n–vector. Then the fourth moments can be put into a square matrix of dimensions $n^2 \times n^2$ in a lot of different ways. In this paper we propose a fourth moment matrix which has nice properties when calculations involving fourth moments have to be carried out such as the correlation between two quadratic forms in x. Furthermore it is shown how to calculate this matrix from the distribution of x by introducing a suitable differential operator acting on the characteristic function.

Keywords characteristic functions, vector differentiation, Kronecker product, permutation matrices.

1. Introduction

Let $x=(x_1,\ldots,x_n)^T$ be a real random n–vector with characteristic function φ which is four times differentiable. Then then all fourth moments of x are finite and are given by

$$E(x_j x_k x_l x_m) = \left[\frac{\partial^4 \varphi(u)}{\partial u_j \partial u_k \partial u_l \partial u_m}\right]_{u=0} \qquad j,k,l,m \in \{1,\ldots,n\}$$

where $u=(u_1,\ldots,u_n)^T \in \mathbb{R}^n$. Clearly any arrangement of these moments into a matrix corresponds to an arrangement of fourth order partial derivatives of the characteristic function φ. Therefore we consider first vector differentiation of arbitrary matrix valued functions.

Let

$$\psi : \mathbf{C}^n \to \mathbf{C}^m \qquad \text{and} \qquad \Psi : \mathbf{C}^{n\times q} \to \mathbf{C}^{m\times p}$$

be differentiable funtions where $\mathbf{C}^{n\times q}$ represents the space of complex $n\times q$ matrices. The usual way of putting their partial derivatives into a matrix is

$$\frac{\partial\psi(z)}{\partial z^T} = \begin{bmatrix} \frac{\partial\psi_1(z)}{\partial z_1} & \cdots & \frac{\partial\psi_1(z)}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial\psi_m(z)}{\partial z_1} & \cdots & \frac{\partial\psi_m(z)}{\partial z_n} \end{bmatrix} \qquad \text{and} \qquad \mathcal{D}\Psi(Z) = \frac{\partial\ vec\Psi(Z)}{\partial(vecZ)^T}\ .$$

According to Magnus and Neudecker (1988) this is the only good way to do this and indeed, when differentials are to be calculated it is the only method that makes sense. However, in our case we are dealing with repeated differentiation with respect to a vector rather than a matrix and we are merely interested in the values at zero rather than in differentials. Furthermore we want an arrangement of fourth order partial derivatives in a matrix in such a way that it corresponds to a matrix of fourth moments which is suitable for matrix manipulations. A matrix containing all fourth moments with nice properties is

$$Q = E\{xx^T \otimes xx^T\}$$

where $\otimes$ stands for the Kronecker product. Two well known and frequently used properties of Kronecker products are

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$$

$$vec(AXC) = (C^T\otimes A)vec(X) .$$

We shall refer to the matrix Q as the *Quadruple matrix*. As an example of its use we give an expression for the product moment of two quadratic forms in terms of Q

$$E(x^TAx)(x^TBx) = \mathrm{tr}\{(A\otimes B)Q\} .$$

Proof. The result follows immediately from the observation that

$$\begin{aligned}(x^TAx)(x^TBx) &= \mathrm{tr}[(x^TAx)\otimes(x^TBx)] = \mathrm{tr}[(x^T\otimes x^T)(A\otimes B)(x\otimes x)] = \\ &= \mathrm{tr}[(A\otimes B)(x^T\otimes x^T)(x\otimes x)]\end{aligned}$$ □

In the next sections we develop a differential operator that enables us to calculate the quadruple matrix from the characteristic function and give an application.

2. Differential Operators

In this section we consider the class of complex matrix valued functions of a vector $u\in\mathbb{C}^n$ which are component wise partially differentiable and define the differential operators $D_\otimes$ and $D_\otimes^T$ by

$$D_\otimes\varphi(u) = \begin{bmatrix} \frac{\partial\varphi(u)}{\partial u_1} \\ \vdots \\ \frac{\partial\varphi(u)}{\partial u_n} \end{bmatrix} \qquad \text{and} \qquad D_\otimes^T\varphi(u) = (D_\otimes\varphi(u)^T)^T .$$

When the argument vector u does not play a role we shall shortly write $D_\otimes\varphi$.

The symbols $D_\otimes$ and $D_\otimes^T$ are motivated by the fact that $D_\otimes\varphi(u)$ looks like the Kronecker product of the operator vector $(\partial/\partial u_1,\ldots,\partial/\partial u_n)^T$ and the matrix $\varphi(u)$. Clearly we have $D_\otimes u^T = I_n$ (the nxn unit matrix) and $D_\otimes u = vecI_n$. The operators turn out to be an adequate tool in particular when Kronecker products are involved. In particular the Quadruple matrix can now be written as

$$Q = \left[D_\otimes^T D_\otimes D_\otimes^T D_\otimes \varphi(u)\right]_{u=0} .$$

Before we can derive the rules of calculation for the differential operators we have to introduce some notation and list some properties of permutation matrices.

3. Permutation Matrices

Let $e_i \in \mathbf{C}^r$ denote the i^{th} unit vector in $\mathbf{C}^r$ $(i=1,\ldots,r)$ and $\hat{e}_j$ the j^{th} unit vector in $\mathbf{C}^p$ $(j=1,\ldots,p)$. Then the permutation matrix $P_{p,r}$ is defined as

$$P_{p,r} = \sum_{i,j} [e_i\hat{e}_j^T \otimes \hat{e}_j e_i^T].$$

Some elementary properties are

$$P_{p,r}P_{r,p} = I_{rp} \qquad P_{p,r}^T = P_{r,p}$$

$$P_{p,1} = I_p \qquad P_{1,r} = I_r$$

but their importance is due to the fact that for arbitrary matrices A $(p\times q)$ and B $(r\times s)$ we have

$$A\otimes B = P_{r,p}(B\otimes A)P_{q,s} .$$

4. Rules of calculation

In the following A and B are arbitrary constant matrices and φ and ψ are arbitrary differentiable matrix valued functions of $u \in \mathbf{C}^n$. When necessary their dimensions are given explicitly, otherwise they follow from the context. We have

1) $D_{\otimes}(A\varphi B) = (I_n \otimes A)(D_{\otimes}\varphi)B$

2) $D_{\otimes}(\varphi \otimes B) = (D_{\otimes}\varphi) \otimes B$

3) When $A \in \mathbf{C}^{p \times q}$ and $\varphi \in \mathbf{C}^{r \times s}$ then $D_{\otimes}(A \otimes \varphi) = [D_{\otimes}(I_p \otimes \varphi)](A \otimes I_s)$

4) *Product rule*

$$D_{\otimes}(\varphi\psi) = (D_{\otimes}\varphi)\psi + (I_n \otimes \varphi)D_{\otimes}\psi$$

5) *Chain rule for scalar φ and $\psi \in \mathbf{C}^m$*

$$D_{\otimes}(\varphi \circ \psi) = (D_{\otimes}\psi^T)(D_{\otimes}^{\psi}\varphi)$$

where $D_{\otimes}^{\psi}$ denotes differentiation w.r.t. ψ.

So far the proofs are straight forward by looking at the components. As an example of their use consider an arbitrary nxn matrix A. Then we have

$$D_{\otimes}(u^T A u) \overset{4}{=} (D_{\otimes}u^T A)u + (I_n \otimes u^T A)D_{\otimes}u \overset{1}{=} (D_{\otimes}u^T)Au + (I_n \otimes u^T A)vecI_n =$$

$$= Au + vec(u^T A) = Au + A^T u \ .$$

In order to be able to differentiate all kinds of functions we need one more rule

6) $D_{\otimes}(I_p \otimes u^T) = P_{p,n}$

The proof follows immediately from 1) and the given properties of permutation matrices i.e.

$$D_{\otimes}(I_p \otimes u^T) = D_{\otimes}[P_{1,p}(u^T \otimes I_p)P_{p,n}] = [D_{\otimes}(u^T \otimes I_p)]P_{p,n} = [I_n \otimes I_p]P_{p,n} = P_{p,n} \qquad \square$$

We are now in a position that we can calculate the quadruple matrix from the characteristic function. As an illustration we calculate in the next section the quadruple matrix for the multivariate normal distribution.

5. The Normal case

Suppose the random vector x has the n-variate normal distribution with dispersion matrix Σ and expectation zero i.e. $x \sim \mathcal{N}_n(0,\Sigma)$. (It is allowed for Σ to be singular). The characteristic function φ is given by

$$\varphi(u) = e^{-\frac{1}{2}u^T \Sigma u} \qquad u \in \mathbb{R}^n.$$

Applying the operator $D_\otimes^T D_\otimes D_\otimes^T D_\otimes$ and taking u=0 yields after some calculations the following nice expression for the Quadruple matrix

$$Q_N = (vec\Sigma)(vec\Sigma)^T + \Sigma\otimes\Sigma + (I_n\otimes\Sigma)P_{n,n}(I_n\otimes\Sigma).$$

6. Application to Wishart matrices

Let Y be a n×n random matrix with the Wishart distribution $W_{m,n}(\Sigma)$ with m degrees of freedom and dispersion matrix Σ. Then expectations of quadratic forms in Y such as $\Psi(V)=E(YVY^T)$ can simply be expressed in the quadruple matrix Q_N of the underlying multinormal distribution. We have the following result

$$vec[\Psi(V)] = [mQ_N + m(m-1)\Sigma\otimes\Sigma]vec(V).$$

The proof is simple and straight forward and left to the reader.

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